JOURNAL OF FUNCTIONAL ANALYSIS 31, 321–332 (1979)

Superexponential Decay of Solutions of a Semilinear Wave Equation*

JENG-ENG LIN

Institute of Applied Mathematics, National Tsing Hua University, Hsinchu, Taiwan, 300, Republic of China

Communicated by the Editors

Received September 8, 1977

Consider a smooth solution of $u_{tt} - \Delta u + q(x) \mid u \mid^{p-1} u = 0$ $x \in R^3$, $q \geqslant 0$ and is C^1, and $1 < p < 5$. Assume that the initial data decay sufficiently rapidly at infinity, $q(x) \leqslant a \exp(-b \mid x \mid^c)$, $a, b > 0$, $c > 1$, and for simplicity, $q_r \leqslant 0$. Then the local energy decays faster than exponentially.

1. INTRODUCTION

It is well known that the wave equation with space dimension odd and $\geqslant 3$ satisfies Huygens' Principle; therefore, if the initial data are of compact support, the local energy of the solution becomes zero after finite time. In the present work, we show that the local energy of the solution of the equation

$$u_{tt} - \Delta u + F(x, u) = 0, \qquad F(x, u)u \geqslant 0, \qquad F(x, 0) = 0, \tag{1.1}$$

in three-space, decays very rapidly ("superexponentially") as $t \to \infty$ under the following conditions: F satisfies a growth restriction on u, $F(x, u)u \geqslant (1 + p) \int_0^u F(x, \omega)\, d\omega$, $p > 1$, and both F as a function of x and initial data decrease superexponentially.

Existence, uniqueness, and smoothness of classical solutions of (1.1) with nice initial data are well known and can be found in Jörgens [2], Segal [7], and Strauss [8]. Strauss [9] proved that for the space dimension odd and $\geqslant 3$, if both F as a function of x and the initial data have compact support and F also satisfies certain conditions, then the local energy of the solution decays exponentially for large time. For linear case, see Lax and Phillips [3], and Morawetz [4]. Furthermore, Strauss [10] proved that, in 3-space, if F is non-

* This paper was a part of the author's Ph.D. dissertation written under the direction of Professor Walter A. Strauss at Brown University, 1976. It was supported in part by NSF Grant MCS 75-08827. The author wishes to thank the referee for suggesting this title.

0022-1236/79/030321-12$02.00/0

linear in u, then the local energy of the solution actually decays superexponentially for large time. The present work is an extension of Strauss' results to the case that neither F as a function of x nor the initial data are of compact support.

In Section 2, we first give the energy estimates (Theorem 1) following Morawetz [5], Strauss [11], and Cooper and Strauss [1]. We then state the exponential decay result of Strauss [9] in Theorem 2. We also repeat the proof of the superexponential decay of Strauss [10] in Theorem 3. In Section 3, we give our main result in Theorem 4: Consider a smooth solution of (1.1) in 3-space, where, for example, $F(x, u) = q(x) \mid u \mid^{p-1} u$, $1 < p < 5$, $q \geqslant 0$, $q \in C^1$, and q decreases superexponentially. If the initial data also decay superexponentially, then the local energy decays superexponentially for large time. The idea of the proof is to use "cut-off functions" on both F and initial data, estimate the energy of the difference between the original solution and the "cut-off solutions," and then apply the result of Strauss [10] on the local energy of the "cut-off solutions." It is interesting to note that our method only uses elementary properties of hyperbolic equations, advanced calculus, and functional analysis. All the functions in this work are real-valued. Also we need only work on the case $t \geqslant 0$.

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We shall use the following standard notations: R^k is the k-dimensional Euclidean space, k is a positive integer. Let Ω be a domain in R^k, $L^p(\Omega)$ denotes the usual Lebesque space with the norm $\mid f \mid_{L^p(\Omega)} = \{\int_\Omega \mid f(x)\mid^p dx\}^{1/p}$, $1 \leqslant p < \infty$, $\mid f \mid_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} \mid f(x)\mid$, $H^m(\Omega)$ denotes the Sobolev space with the norm $\mid f \mid_{H^m(\Omega)} = \{\sum_{|\alpha|\leqslant m} \int_\Omega \mid D^\alpha f(x)\mid^2 dx\}^{1/2}$, where m is a nonnegative integer, and α is a multi-index. $C^j(\Omega)$ denotes the space of functions which have continuously partial derivatives of order up to and including j (of order $< \infty$ if $j = \infty$) in Ω. $C_0{}^j(\Omega)$ denotes the set of all functions $\in C^j(\Omega)$ with compact support in Ω. We shall write $\mid f \mid_p = \mid f \mid_{L^p(R^k)}$ and $\mid f \mid_{m,2} = \mid f \mid_{H^m(R^k)}$. When the domain of the integration is not mentioned, it is understood that the domain is the whole space. Let F satisfy

$$F \in C^0, \quad F(x, s)s \geqslant 0, \quad F(x, 0) = 0 \qquad \text{for all } x \in R^n, \quad s \in R^1, \qquad \text{(F.1)}$$

then we define $G(x, s) = \int_0^s F(x, \omega)\, d\omega$, for all $x \in R^n$, $s \in R^1$. Note that $G \geqslant 0$.

Define

$$\hat{E}_l[f(t)] = \tfrac{1}{2} \int_{|x|<l} (\mid f_t \mid^2 + \mid \nabla f \mid^2)(x, t)\, dx,$$

$$E_l[f(t)] = \hat{E}_l[f(t)] + \int_{|x|<l} G(x, f(x, t))\, dx,$$

for all $\ell > 0$, $t \geqslant 0$, where $f \in C^1$, and define

$$E[f(t)] = \lim_{l \to \infty} E_l[f(t)].$$

THEOREM 1 (Following [5], [11], and [1]). *Consider a C^2 solution $u(x, t)$ of* (1.1), $x \in R^n$, $n \geqslant 3$. *Assume that* $E[u(0)] < \infty$ *and F satisfies* (F.2):

$$\text{there exist } \zeta \in C^2(R^n),\ k \in C^1(R^n), \text{ and constant } p_1 > 1, \tag{F.2}$$

such that

$$0 \leqslant \zeta \leqslant 1, \qquad 0 \leqslant k, \qquad 0 < \zeta_r \leqslant \frac{1}{r}\zeta,$$

$$\zeta_r \mid \theta_r \mid^2 \leqslant (\nabla\theta \cdot \nabla\zeta)\,\theta_r\,, \qquad \text{for all } \theta \in C^1(R^n),$$

$$2r\zeta_r - r^2 \Delta\zeta + (n-3)\zeta \geqslant 0, \qquad F(x, s)s \geqslant (p_1 + 1)\, G(x, s),$$

$$(n-1)(p_1 - 1)\zeta \geqslant 2r\left(\frac{1}{k} + \zeta_r\right),$$

and

$$G_r(x, s) \leqslant (2r\zeta)^{-1}\left[(n-1)(p_1 - 1)\,\zeta(x) - 2r\left(\frac{1}{k} + \zeta_r\right)(x)\right] G(x, s).$$

for all $x \in R^n$, $s \in R^1$, where $r = \mid x \mid$.

Then

$$\int_0^\infty E_R[u(t)]\, dt \leqslant C_R E[u(0)] \qquad \text{for all } R > 0,$$

where

$$C_R = 2\left(\min_{|x| \leqslant R} \left\{\zeta_r(x), \frac{1}{k(x)}\right\}\right)^{-1}$$

Proof.

$$\begin{aligned}
0 &= (u_{tt} - \Delta u + F(x, u))\left(u_t + \zeta u_r + \frac{n-1}{2r}\, u\right) \\
&= \frac{\partial X}{\partial t} + \nabla \cdot Y + Z + \frac{\partial}{\partial t}(G(x, u)) \\
&\quad + \nabla \cdot \left[\zeta \frac{x}{r} G(x, u)\right] - \zeta_r G(x, u) - \frac{(n-1)}{r}\,\zeta G(x, u) \\
&\quad - \zeta G_r(x, u) + \zeta\,\frac{n-1}{2r} F(x, u) u,
\end{aligned}$$

where

$$X = \tfrac{1}{2}u_t^2 + \zeta u_r u_t + \zeta \frac{n-1}{2r} u u_t + \tfrac{1}{2} | \nabla u |^2,$$

$$Y = -\zeta \frac{x}{2r} u_t^2 - (\nabla u) \left(u_t + \zeta u_r + \zeta \frac{n-1}{2r} u \right) + \zeta \frac{x}{2r} | \nabla u |^2$$

$$+ (\nabla \zeta) \frac{n-1}{4r} u^2 - \zeta \frac{(n-1)x}{4r^3} u^2,$$

$$Z = \tfrac{1}{2}\zeta_r(u_t^2 + | \nabla u |^2) + (\nabla u \cdot \nabla \zeta) u_r - \zeta_r | u_r |^2$$

$$+ \left(\frac{1}{r} \zeta - \zeta_r \right) (| \nabla u |^2 - | u_r |^2)$$

$$+ 4^{-1} r^{-3} (n-1)(2r\zeta_r - r^2 \Delta\zeta + (n-3)\zeta) u^2$$

Moreover,

$$X = \tfrac{1}{2}(1 - \zeta)(u_t^2 + | \nabla u |^2) + \zeta \left[\tfrac{1}{2}u_t^2 + \tfrac{1}{2} | w |^2 + \left(\frac{x}{r} \cdot w \right) u_t \right]$$

$$- \nabla \cdot \left[\frac{(n-1)x}{4r^2} \zeta u^2 \right] + \frac{(n-1)(n-3)}{8r^2} \zeta u^2 + \frac{n-1}{4r} \zeta_r u^2,$$

and

$$X = \tfrac{1}{2}(1 + \zeta)(u_t^2 + | \nabla u |^2) - \zeta \left[\tfrac{1}{2}u_t^2 + \tfrac{1}{2} | w |^2 - \left(\frac{x}{r} \cdot w \right) u_t \right]$$

$$+ \nabla \cdot \left[\frac{(n-1)x}{4r^2} \zeta u^2 \right] - \frac{(n-1)(n-3)}{8r^2} \zeta u^2 - \frac{n-1}{4r} \zeta_r u^2,$$

where

$$w = \nabla u + \frac{n-1}{2} \frac{x}{r^2} u.$$

Hence,

$$0 \geqslant \frac{\partial X}{\partial t} + \nabla \cdot Y + \tfrac{1}{2}\zeta_r(u_t^2 + | \nabla u |^2) + \frac{\partial}{\partial t} G(x, u)$$

$$+ \nabla \cdot \left[\zeta \frac{x}{r} G(x, u) \right] + \frac{1}{k(x)} G(x, u)$$

Hence,

$$0 \geqslant \int_{|x|<\infty} X(x, T)\, dx - \int_{|x|<\infty} X(x, 0)\, dx$$

$$+ \int_{|x|<\infty} G(x, u(x, T))\, dx - \int_{|x|<\infty} G(x, u(x, 0))\, dx$$

$$+ \int_0^T \int_{|x|<\infty} \tfrac{1}{2}\zeta_r(u_t^2 + | \nabla u |^2)(x, t)\, dx\, dt$$

$$+ \int_0^T \int_{|x|<\infty} \frac{1}{k(x)} G(x, u(x, t))\, dx\, dt$$

Hence,

$$\int_{|x|<\infty} [\tfrac{1}{2}(1 + \zeta)(u_t^2 + |\nabla u|^2)(x, 0) + G(x, u(x, 0))]\, dx$$
$$\geqslant \left(\int_0^T \int_{|x|\leqslant R} [\tfrac{1}{2}(u_t^2 + |\nabla u|^2)(x, t) + G(x, u(x, t))]\, dx\, dt\right)$$
$$\cdot \min_{|x|\leqslant R} \left\{\zeta_r(x), \frac{1}{k(x)}\right\}.$$

Let

$$C_R = 2\left(\min_{|x|\leqslant R} \left\{\zeta_r(x), \frac{1}{k(x)}\right\}\right)^{-1},$$

then

$$\int_0^\infty E_R[u(t)]\, dt \leqslant C_R E[u(0)].$$

COROLLARY 1. *Under the assumptions of Theorem* 1, *given* $\delta > 0$, *for all* C^2 *solutions* $u(x, t)$ *with initial energy* $E[u(0)] < \infty$, *there is a time* T *such that* $E_R[u(T)] \leqslant \delta E[u(0)]$, $0 \leqslant T \leqslant C_R \delta^{-1}$, *where* C_R *is as in Theorem* 1.

THEOREM 2 (Exponential Decay) ([9]). *Let* $u(x, t)$ *be a* C^2 *solution of* (1.1), $x \in R^n$, $n \geqslant 3$ *and* n *is odd. Assume that* F *satisfies* (F.1), (F.2), *and* (F.I): *both* F *as a function of* x *and the initial data have compact support* $\subset \{x \mid |x| < \rho\}$ *for some* $\rho > 1$. *Let* u *satisfy*

$$u(t) \in H_{\mathrm{loc}}^{[n/2]+3} \qquad \textit{and} \qquad u_t(t) \in H_{\mathrm{loc}}^{[n/2]+2} \tag{I.1}$$

for all $t \geqslant 0$.

Given $r > 0$, then $E_r[u(t)] \leqslant 8E[u(0)] \exp(-(t - r - 2\rho)\, \ell n 2/(8C_{5\rho} + 6\rho))$ for all $t > r + 2\rho$, where $C_{5\rho}$ is C_R in Theorem 1 with $R = 5\rho$.

The space dimension is *three* in the rest of this paper. C will denote various constants in the rest of this paper.

LEMMA 1 (Cf. Lemma 1.1 of [8]). *Consider* $u_{0tt} - \Delta u_0 = 0$, $u_0(x, 0) = 0$, $u_{0t}(x, 0) = \phi(x)$, $\phi \in L^2 \cap L^{6/5}$, $x \in R^3$. *Then* $|u_0|_2(t) \leqslant C |\phi|_{6/5}$ *for some absolute constant* $C > 0$ *and for all* $t \geqslant 0$.

Proof. We need only prove the case $\phi \in C_0^\infty$. Define R_j and ψ through Fourier transform by

$$\widehat{R_j\phi}(\xi) = |\xi|^{-1} \xi_j \hat{\phi}(\xi),$$
$$j = 1, 2, 3,$$

and

$$|\xi|\, \hat{\psi}(\xi) = \hat{\phi}(\xi).$$

Then we have

$$
\begin{aligned}
|u_0|_2(t) &= |\hat{u}_0|_2(t) \leqslant \| \cdot |^{-1}\hat{\phi}(\cdot)|_2 = |\hat{\psi}|_2 \\
&= |\psi|_2 \leqslant C \sum_{j=1}^{3} \left| \frac{\partial \psi}{\partial x_j} \right|_{6/5} \\
&= C \sum_{j=1}^{3} | R_j \phi |_{6/5} \leqslant C | \phi |_{6/5} .
\end{aligned}
$$

LEMMA 2 (Uniform Boundedness). *Consider a C^2 solution $u(x, t)$ of* (1.1) *with space dimension three. Assume that the initial data satisfy*

$$\phi \in L^\infty \cap H^2 \cap C^2,\ |\nabla\phi| \in L^\infty,\ \text{and } \psi \in L^\infty \cap H^1 \cap C^0 \text{ and } F \tag{I.2}$$

satisfies

$$|F(x, s)| \leqslant \alpha_0 (G(x, s))^{-\epsilon_2 + 2/3} |s|^{-\epsilon_1 + 1} \tag{F.3}$$

for all $x \in R^3$, $s \in R^1$, where $\alpha_0 > 0$, $0 < \epsilon_2 < 2/3$, $0 < \epsilon_1 < 1$ are constants, and (F.4): *there exists $f \in L^1$ such that*

$$|F(x, s)| \leqslant \alpha_1(|f(x)| + G(x, s)), \tag{F.4}$$

for all $x \in R^3$, $s \in R^1$, where $\alpha_1 > 0$ is a constant.

Then u is uniformly bounded in all space-time. The uniform bound depends only on $|\phi|_\infty$, $|\nabla\phi|_\infty$, $|\psi|_\infty$, $|\phi|_{2,2}$, $|\psi|_{1,2}$, α_0, α_1, ϵ_1, ϵ_2, $E[u(0)]$, $|f|_1$, and some absolute constants.

The proof is following the proof of Lemma 6 of [6].

LEMMA 3. *Let f be a real-valued continuous function on $[0, \infty)$. Assume that*

$$0 \leqslant f(t) \leqslant a_1 \exp(-a_2 t)$$

for all $t > a_3$, where $a_1, a_2, a_3 > 0$ are constants, and

$$f(t) \leqslant a_4 \int_{t-a_5}^{t} (f(\tau))^{a_6}\, d\tau,$$

for all $t > a_3 + a_5$, where $a_4, a_5 > 0$, and $a_6 > 1$ are constants. We may assume that $a_4 a_5 > 1$. Let δ_0 and δ be such that $0 < \delta_0 < 1 - \delta < 1$.

Then

$$f(t) \leqslant \exp(-a_6^{t^\delta}),$$

if $t > t^{1-\delta} \geqslant t^{\delta_0} + 2a_5$, $t^{\delta_0} > a_3$, and

$$a_1 (a_4 a_5)^{(a_6 - 1)^{-1}} \exp(-a_2 t^{\delta_0}) < e^{-1}.$$

The proof of this lemma is by an elementary iteration method.

THEOREM 3 (Superexponential Decay) (Following [10]). *Consider a C^2 solution of* (1.1) *with space dimension three. Assume that* (F.1), (F.2), (F.3), (F.4), (F.I), (I.1), *and* (I.2) *are satisfied. Assume that F also satisfies*

$$|F(x, s)| \leqslant \alpha_2 |s|^p \tag{$*$}$$

for all $x \in R^3$, $s \in R^1$, where $\alpha_2 > 0$ and $P > 1$ are constants. Let δ and δ_0 be constants such that $0 < \delta_0 < 1 - \delta < 1$.

Then

$$E_{2\rho}[u(t)] \leqslant \exp(-a_6^{t^\delta}),$$

if $t > t^{1-\delta} \geqslant t^{\delta_0} + 2a_5$, $t^{\delta_0} > a_3$, *and*

$$a_1(a_4 a_5)^{(a_6-1)^{-1}} \exp(-a_2 t^{\delta_0}) < e^{-1},$$

where

$$\begin{aligned}
a_1 &= \beta_1(E[u(0)] + 1)\,\rho^{7-3\epsilon_3}, \\
a_2 &= \epsilon_3 \ln 2/(8C_{5\rho} + 6\rho), \\
a_3 &= 6\rho, \qquad a_4 = \beta_2 \rho^{7/3} && \text{if } p \geqslant 5/3, \\
a_4 &= \beta_2' \rho^{9-4p} && \text{if } 1 < p < 5/3, \\
a_5 &= 3\rho, \qquad a_6 = 5/3, && \text{if } p \geqslant 5/3, \\
a_6 &= p, && \text{if } 1 < p < 5/3,
\end{aligned}$$

ϵ_3 *is any given constant such that*

$$\max\{0, 1/3 - 2\epsilon_2\} < \epsilon_3 < \min\{1, 4/3 - 2\epsilon_2\}$$

where ϵ_2 is from (F.3), *$\beta_1 > 0$, depends only on N, p, α_0, α_2, ϵ_1, ϵ_2, ϵ_3, and some absolute constants, and β_2 and $\beta_2' > 1$ depend only on N, p, α_2, and some absolute constants, where N is any given uniform bound of u.*

Proof. We can write

$$u(x, t) = u_0(x, t) - \int_0^t R(t - \tau) * F(\cdot, u(\cdot, \tau))\, d\tau$$

where

$$R(x, t) = \delta(|x| - t)/4\pi t$$

is the Riemann function of the wave equation, and u_0 is the solution of the wave equation with the initial data at $t = 0$ same as those of u.

Since both F as a function of x and initial data have compact support $\subset \{x \mid |x| < \rho\}$, by Huygens' Principle,

$$u(x, t) = -\int_{t-3\rho}^{t} R(t-\tau) * F(\cdot, u(\cdot, \tau))\, d\tau \tag{1.2}$$

for $t > 3\rho$, $|x| < 2\rho$.

For simplicity, we assume $p \geqslant 5/3$.

From Lemma 1, we have

$$|u|^2_{L^2(|x|<2\rho)}(t) \leqslant C\rho \int_{t-3\rho}^{t} |u|^{10/3}_{L^2(|x|<\rho)}(\tau)\, d\tau$$

From $(*)$,

$$\int_{|x|<2\rho} G(x, u(x, t))\, dx \leqslant C\, |u|^2_{L^2(|x|<2\rho)}(t),$$

Define

$$H_r[u(t)] = E_r[u(t)] + \int_{|x|<r} |u|^2(x, t)\, dx, \qquad \text{for all } r > 0.$$

Hence

$$H_{2\rho}[u(t)] \leqslant C\rho \int_{t-3\rho}^{t} (\mathring{E}_{2\rho}[R(t-\tau) * F(\cdot, u(\cdot, \tau))] + |u|^{10/3}_{L^2(|x|<\rho)}(\tau))\, d\tau$$

On the other hand, by Sobolev inequality,

$$\mathring{E}_{2\rho}[R(t-\tau) * F(\cdot, u(\cdot, \tau))] \leqslant C\rho^{4/3}(|u|^2_{L^2(|x|<2\rho)}(\tau) + |\nabla u|^2_{L^2(|x|<2\rho)}(\tau))^{5/3}$$

Hence

$$H_{2\rho}[u(t)] \leqslant \beta_2 \rho^{7/3} \int_{t-3\rho}^{t} (H_{2\rho}[u(\tau)])^{5/3}\, d\tau. \tag{A}$$

Since local energy of u decays exponentially and u is uniformly bounded in all space-time, therefore we may expect to prove that local L^2 norm of u also decays exponentially. In fact, from (1.2) and (F.3), we have

$$|u|^2_{L^2(|x|<2\rho)}(t) \leqslant C\rho^{6-3\epsilon_3} \int_{t-3\rho}^{t} E_\rho[u(\tau)]^{\epsilon_3}\, d\tau.$$

Hence we have exponential decay of local L^2 norm of u. Therefore, by Theorem 2,

$$H_{2\rho}[u(t)] \leqslant \beta_1(E[u(0)] + 1)\, \rho^{7-3\epsilon_3} \exp(-t\epsilon_3 \ln 2/(8C_{5\rho} + 6\rho)), \quad \text{for } t > 6\rho. \tag{B}$$

From (A) and (B), the conclusion of this theorem follows from Lemma 3

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THEOREM 4 (Main Theorem). *Consider a C^2 solution $u(x, t)$ of* (1.1) *with space dimension three.*

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x).$$

Assume (1): (F.1), (F.2), (F.3), (F.4), (I.1), *and* (I.2) *are satisfied,*

(2): *there is a continuous function $q \geqslant 0$ such that*

$$| F(x, s_1) - F(x, s_2)| \leqslant 2q(x) \mid s_1 - s_2 \mid (\mid s_1 \mid^{p-1} + \mid s_2 \mid^{p-1})$$

for all $x \in R^3$, $s_1, s_2 \in R^1$, where $p > 1$ and

$$0 \leqslant q(x) \leqslant b_0 \exp(-w_0 \mid x \mid^{h_0}), \tag{$**$}$$

for some constants $w_0, b_0 > 0$, $h_0 > 1$, and for all $x \in R^3$,

(3): *there is a constant $a \geqslant 2$ such that*

$$G(x, s_1 + s_2) \leqslant a(G(x, s_1) + G(x, s_2))$$

for all $x \in R^3$, $s_1, s_2 \in R^1$,

(4): $\sup_{|x| \geqslant T}\{| \psi |(x), | \phi |(x), | \nabla\phi |(x)\} \leqslant b \exp(-wT^h)$ *for some constants $b, w > 0$, $h > 1$, and for all $T \geqslant 0$,*

(5): *there exist constants ϵ, $a_\epsilon > 0$ such that $C_R \leqslant a_\epsilon R^{1+\epsilon}$ for all $R \geqslant 1$, where C_R is from Theorem* 1, *and*

$$0 < \epsilon < \min\{h - 1, h_0 - 1\}.$$

Let $r > 0$ and let $\delta_1 > 0$ such that

$$(\min\{h, h_0\})^{-1} < \delta_1 < (1 + \epsilon)^{-1}.$$

Then

$$E_r[u(t)] \leqslant \alpha \exp(-\beta t^\lambda), \qquad \alpha, \beta > 0, \quad \lambda > 1,$$

if t is large enough, where $\beta = \min\{w/2, w_0/4\}$, and $\lambda = \delta_1 \min\{h, h_0\}$. The dependence of α and the size of t can be found in the proof.

Proof. Let $g^T \in C_0^\infty(R^3)$, $T > 1$, such that $g^T(x) = 1$, if $| x | \leqslant T$, $g^T(x) = 0$ if $| x | \geqslant 2T$, $0 \leqslant g^T \leqslant 1$, and $\sum_{m \leqslant 2} | D^m g^T | < M$ for some constant $M > 0$ which is independent of T. Then, by Lemma 2, there exists $N > 0$, which is independent of T, such that $| u | < N$ and $| u^T | < N$, for all $T > 1$, where u^T is any C^2 solution of

$$\begin{gathered} u_{tt}^T - \Delta u^T + g^T F(x, u^T) = 0 \\ u^T(x, 0) = g^T\phi(x), \qquad u_t^T(x, 0) = g^T\psi(x). \end{gathered} \tag{1.3}$$

Define

$$E_r^T[f(t)] = \int_{|x|<r} [\tfrac{1}{2}(|f_t|^2 + |\nabla f|^2)(x, t) + g^T(x)\, G(x, f(x, t))]\, dx$$

for all $f \in C^1$ and $r > 0$. Substracting (1.3) from (1.1), multiplying the result equation by $(u - u^T)_t$, and integrating over $R^3 \times [0, t]$, we have

$$E^T[(u - u^T)(t)] \leqslant E^T[(u - u^T)(0)] + 2^{p+1}N^{p+1} \int_0^t \int q\, |u - u^T|\, |(u - u^T)_t|\, dx\, d\tau$$
$$+ N^p\, |q|^{1/2}_{L^\infty(|x|\geqslant T)} \int_0^t \int q^{1/2}\, |(u - u^T)_t|\, dx\, d\tau.$$

Now,

$$\int q\, |u - u^T|\, |(u - u^T)_t|(x, \tau)\, dx$$
$$\leqslant \tfrac{1}{2}\, |q|_{3/2}\, |\, |u - u^T|^2\, |_3(\tau) + |q|_\infty\, E^T[(u - u^T)(\tau)]$$
$$\leqslant CE^T[(u - u^T)(\tau)],$$

and $\int q^{1/2}\, |(u - u^T)_t|(x, \tau)\, dx$ is bounded by a constant which depends only on $|q|_1$, M, and $E[u(0)]$.

Therefore,

$$E^T[(u - u^T)(t)] \leqslant E^T[(u - u^T)(0)] + Ct\, |q|^{1/2}_{L^\infty(|x|\geqslant T)} + \mu \int_0^t E^T[(u - u^T)(\tau)]\, d\tau.$$

Hence

$$E^T[(u - u^T)(t)] \leqslant C \exp(-wT^h) + Ct \exp(-\tfrac{1}{2}w_0 T^{h_0}) + Ct \exp(-wT^h + \mu t)$$
$$+ Ct^2 \exp(-\tfrac{1}{2}w_0 T^{h_0} + \mu t),$$

where $\mu > 0$ is a constant which depends only on N, p, $|q|_{3/2}$, b_0, and an absolute constant, and $C > 0$ is a constant which depends only on $|\psi|_1$, $|\phi|_1$, $|\nabla\phi|_1$, b, b_0, M, N, and p.

Now,

$$E_r[u(t)] = E_r^T[u(t)] \leqslant a(E_r^T[(u - u^T)(t)] + E_r^T[u^T(t)])$$

if $T \geqslant r$.

Let δ_0 and δ be such that $(1 + \epsilon)\,\delta_1 < \delta_0 < 1 - \delta < 1$, and $\epsilon_3 > 0$ be such that

$$\max\{0, 1/3 - 2\epsilon_2\} < \epsilon_3 < \min\{1, 4/3 - 2\epsilon_2\}.$$

For simplicity, assume $p \geqslant 5/3$, then by Theorem 3,

$$E_r^T[u^T(t)] \leqslant \exp(-(5/3)^{t^\delta}),$$

if $t > t^{1-\delta} \geqslant t^{\delta_0} + 12T$, $t^{\delta_0} > 12T \geqslant 3r$, and

$$\beta_3 T^{12} \exp(-t^{\delta_0}\epsilon_3 \ln 2/(8a_\epsilon 10^{1+\epsilon}T^{1+\epsilon} + 12T)) < e^{-1},$$

where $\beta_3 > 0$ is some constant depending only on M, N, α_0, α_2, p, ϵ_1, ϵ_2, ϵ_3, $E[u(0)]$, and some absolute constants.

Let $T = t^{\delta_1}$, then

$$E_r[u(t)] \leqslant \alpha \exp(-\beta t^\lambda) + a \exp(-(5/3)^{t^\delta}),$$

$\alpha, \beta, a, \delta > 0, \lambda > 1$, if

$$t^{\delta_1 h_0} > 4(\mu t + 2 \ln t)/w_0, \qquad t^{\delta_1 h} > 2(\mu t + \ln t)/w,$$

$$t > t^{1-\delta} \geqslant t^{\delta_0} + 12t^{\delta_1}, \qquad t^{\delta_0} > 12t^{\delta_1} \geqslant 12r,$$

$$\beta_3 t^{12\delta_1} \exp(-t^{\delta_0}\epsilon_3 \ln 2/(8a_\epsilon 10^{1+\epsilon}t^{\delta_1(1+\epsilon)} + 12t^{\delta_1})) < e^{-1},$$

where $\alpha > 0$ depends only on $|\psi|_1$, $|\phi|_1$, $|\nabla\phi|_1$, a, b, b_0, M, N, and p, $\beta = \min\{w/2, w_0/4\}$, and $\lambda = \delta_1 \min\{h, h_0\}$.

Remark. If $1 < p < 5$, $q \in C^1$, $q \geqslant 0$, q satisfies $(**)$, and $q_r \leqslant 0$, then $F(x, u) = q(x) | u |^{p-1} u$, $x \in R^3$, satisfies the assumptions of Theorem 4 on F.

EXAMPLE. Let $F(x, u) = q(x) | u |^{p-1} u$, $3 \leqslant p < 5$, where $q \geqslant 0$, $q \in C^3$, q and all its derivatives of order $\leqslant 3$ are uniformly bounded, and $x \in R^3$. Let $\phi \in H^4_{\text{loc}}(R^3)$ and $\psi \in H^3_{\text{loc}}(R^3)$. Then, by Theorem 1.9 of [8], there is a unique solution $u(x, t)$ of (1.1), with initial data $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$, such that u is a C^j function of t with values in $H^{4-j}_{\text{loc}}(R^3)$ for $j = 0, 1, 2$. Therefore, u is C^2. Let ϕ and ψ satisfy the assumption (4) of Theorem 4, and let q satisfy $(**)$ and $-\infty < q_r \leqslant (p - 1 - \epsilon) r^{-1}(1 + r^\epsilon)^{-1} q$, where $0 < \epsilon < \min\{1/2, h - 1, h_0 - 1\}$, h is from the assumption (4) of Theorem 4. Then F with $\zeta(x) = 1 - (1 + | x |^\epsilon)^{-1}$ and $k(x) = r^{1-2\epsilon}(1 + r^\epsilon)^2 (p - 1)^{-1}$, where $r = | x |$, satisfies (F.2) with $p_1 = p$. In this case, $C_R = 2R^{1-\epsilon}(1 + R^\epsilon)^2 \epsilon^{-1}$, for $R \geqslant 1$, satisfies assumption (5) of Theorem 4.

ACKNOWLEDGMENTS

The author thanks Professor Walter Strauss for his continuing help, patience, advice, encouragement and support during the work of this paper. The author also thanks Professor Constantine Dafermos for his many helpful suggestions.

References

1. J. Cooper and W. A. Strauss, "Energy Boundedness and Decay of Waves Reflecting off a Moving Obstacle," Department of Mathematics, University of Maryland, Tech. Rept. TR-74-69, 1974.
2. K. Jörgens, Das Anfangswertproblem in Grossen für eine Klasse nichtlinearer Wellengleichungen, *Math. Z.* **77** (1961), 295–308.
3. P. D. Lax and R. S. Phillips, "Scattering Theory," Academic Press, New York, 1967.
4. C. S. Morawetz, Exponential decay of solutions of the wave equation, *Comm. Pure Appl. Math.* **19** (1966), 439–444.
5. C. S. Morawetz, Time decay for the nonlinear Klein–Gordon equation, *Proc. Royal Soc. London Ser. A* **306** (1968), 291–296.
6. C. S. Morawetz and W. A. Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, *Comm. Pure Appl. Math.* **25** (1972), 1–31.
7. I. Segal, Nonlinear semigroups, *Ann. of Math.* **78** (1963), 330–364.
8. W. A. Strauss, Decay and asymptotics for $\square u = F(u)$, *J. Functional Analysis* **2** (1968), 409–457.
9. W. A. Strauss, Decay of solutions of hyperbolic equations with localized nonlinear terms, *in* "Symposia Mathematica VII, 1971," pp. 339–355.
10. W. A. Strauss, Nonlinear scattering theory, *in* "Scattering Theory in Mathematical Physics," pp. 53–78, Reidel, Dordrecht, 1974.
11. W. A. Strauss, Dispersal of waves vanishing on the boundary of an exterior domain, *Comm. Pure Appl. Math.* **28** (1975), 265–278.